Università degli Studi di Milano

Dipartimento di Tecnologie dell'Informazione

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A branch-and-price algorithm for large-scale planar location problems

Giovanni Righini (righini@dti.unimi.it)

Luca Zaniboni (zaniboni@dti.unimi.it)

The Multi-Weber Problem

- Data:
  - A set of N points $P_1,\ldots,P_N$ in the Euclidean plane. Each point i has a weight w_i.
  - A number p of sources to be located.
- Variables:
  - Locate the sources in the Euclidean plane.
  - Assign each point to its closest source.
- Objective:
  - Minimize the sum of the weighted Euclidean distances between each point and its source.

The mathematical model

- Let (x_S, y_S) be the coordinates of each *source* S.
- Let a_{ij} be the assignment variable of point P_i to source S_j.

$$\min \sum_{i=1}^{N} \sum_{j=1}^{p} a_{ij} w_i \mathrm{D}\left(P_i, S_j\right)$$

Euclidean distance (non-linear)

$$\begin{aligned} s.t. \quad & \sum_{j=1}^{p} a_{ij} = 1 && \forall i = 1, \ldots, N \\ & a_{ij} \in \{0,1\} && \forall i = 1, \ldots, N \quad \forall j = 1, \ldots, p \\ & \left(x_{S_j}, y_{S_j}\right) \in \mathbf{R}^2 && \forall j = 1, \ldots, p \end{aligned}$$

Complexity and state of the art

- The MWP problem is NP-hard.
- Its continuous relaxation is neither convex nor concave.
- Assignment decisions (discrete variables) are more critical than location decisions (continuous variables).

- Kuenne & Soland (1972): B&B, small instances.
- Rosing (1992): set covering, N=30, p=5.
- Chen et al. (1998): d.-c. programming, N large, p=3.
- Krau (1997): B&P, N=1060, p=100.

Set partitioning reformulation

$$\min \sum_{k \in K} c_k z_k$$

$$s.t. \quad \sum_{k \in K} a_{ik} z_k = 1 \qquad \forall i = 1, \ldots, N$$

$$\sum_{k \in K} z_k = p$$

$$z_k \in \{0,1\} \qquad \forall k \in K$$

Binary variable equal to 1 if cluster P_k belongs to the solution

K is the set of all clusters

Coefficient equal to 1 if point i belongs to cluster k

Cost of cluster k

A cluster is a subset of points assigned to the same source.

Set covering reformulation

$$\min \sum_{k \in K'} c_k z_k$$

$$s.t. \sum_{k \in K'} a_{ik} z_k \geq 1 \qquad \forall i = 1, \ldots, N$$

$$-\sum_{k \in K'} z_k \geq -p$$

$$z_k \in \{0,1\} \qquad \forall k \in K'$$

Dual variables are indicated by λ_i

The dual variable is indicated by μ

K' is a restricted set of clusters.

Linear relaxation

$$z_k \geq 0$$

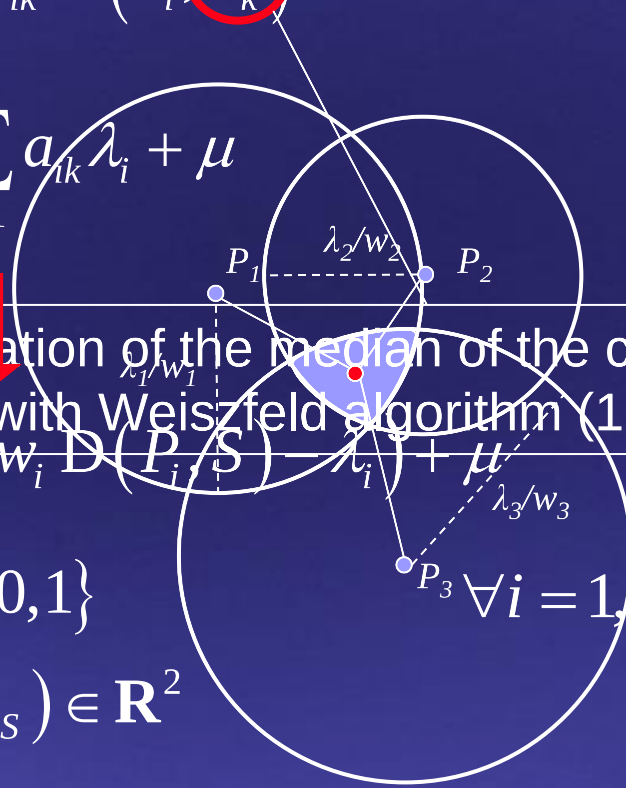
Restricted Linear Master Problem (RLMP)

Pricing (1)

$$c_k = \sum_{i=1}^{N} w_i a_{ik} \, \mathrm{D}\left(P_i, S_k^*\right)$$

$$r_k = c_k - \sum_{i=1}^{N} a_{ik}\lambda_i + \mu$$

Optimal location of the median of the cluster computed with Weiszfeld algorithm (1937).



$$\min \sum_{i=1}^{N} a_i \left(w_i \, \mathrm{D}\left(P_i, S\right) - \lambda_i \right) + \mu$$

$$s.t. \quad a_i \in \{0,1\} \qquad \forall i = 1, \ldots, N$$

$$\left(x_S, y_S\right) \in \mathbf{R}^2$$

Pricing (2)

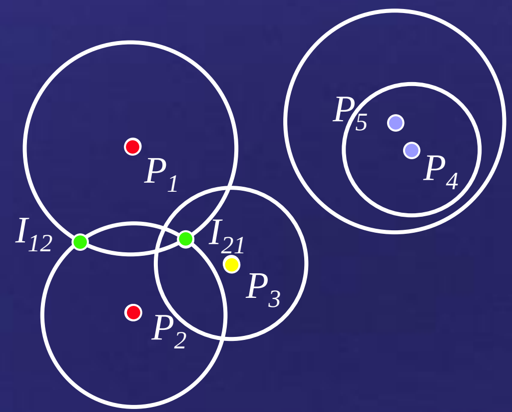
It is not necessary to perform an exhaustive search of all possible subsets of points.

Theorem (Drezner, Mehrez, Wesolowsky, 1991)

The number of distinct regions defined by the intersection of N circles in R^2 is at most $2N(N-1)$.

Pricing (3)

DMW's algorithm:



- Find base points for each pair of circles.
- For each base point find the set of circles covering it.
- For each such set generate 4 clusters.

$$Q=\emptyset\{I_{12}, I_{21}\}$$

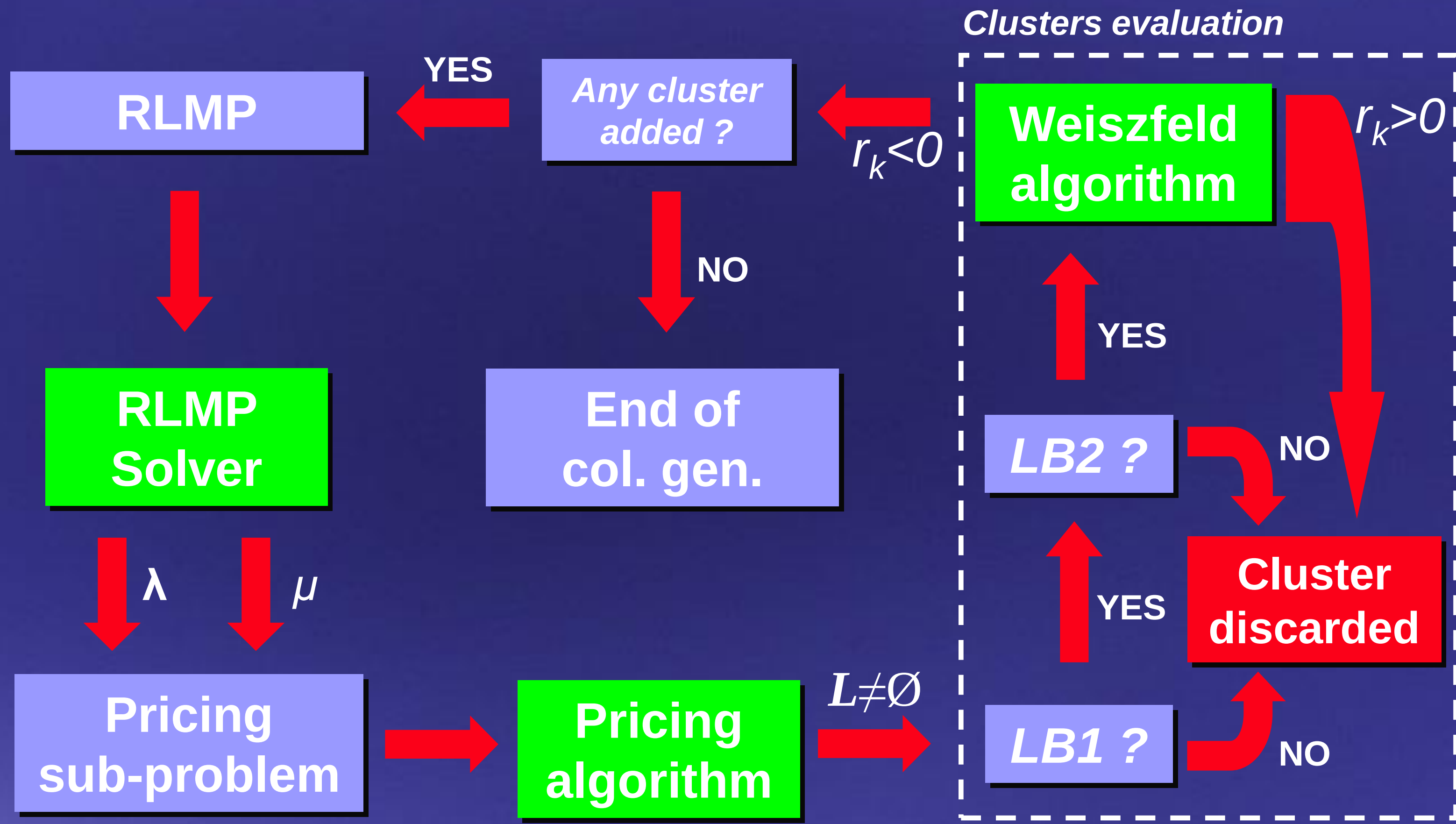
$$C=\emptyset\{P_3\}$$

$$L=\emptyset\{\underbrace{\{P_1, P_3\}}_{C\cup\{P_1\}}, \underbrace{\{P_2, P_3\}}_{C\cup\{P_2\}}, \underbrace{\{P_1, P_2, P_3\}}_{C\cup\{P_1, P_2\}}, \underbrace{\{P_3\}}_{C}\}$$

Pricing (4)

- The drawback of DMW's algorithm is that each cluster can be generated many times.
- This is more evident when clusters are large, that is for small values of *p/N*.
- Hence this pricing method is particularly suitable for MWP instances with large *p/N*.

Column generation (1)



Column generation (2)

Columns management:

- Max n. of columns in the RLMP = 100K
- Removed columns are stored in a pool

Initialization:

- One dummy column covering all the points.
- Many columns covering 1, 2 and 3 points.
- Columns from the most recently solved RLMP.
- Columns taken from the pool.

Column generation (3)

Heuristics:

- Location-allocation algorithm (Cooper, 1963) – 50 runs: it produces many different columns.
- VNS algorithm (Brimberg et al., 2000) – 5 runs: it produces very good columns.

Stabilization:

- Box stabilization (du Merle et al., 1999).
- Interior point stabilization (Rousseau et al., 2003).

Lower bound 1

- This lower bound is due to Drezner (1984).
  Let P_0 be any point in $\mathbb{R}^2$ (e.g. the barycenter of the cluster).

$$w_i^x = \frac{w_i \left| x_0 - x_i \right|}{\mathrm{D}\left(P_i, P_0\right)} \qquad w_i^y = \frac{w_i \left| y_0 - y_i \right|}{\mathrm{D}\left(P_i, P_0\right)}$$

- The following inequality provides a valid lower bound *LB1* to r_c:

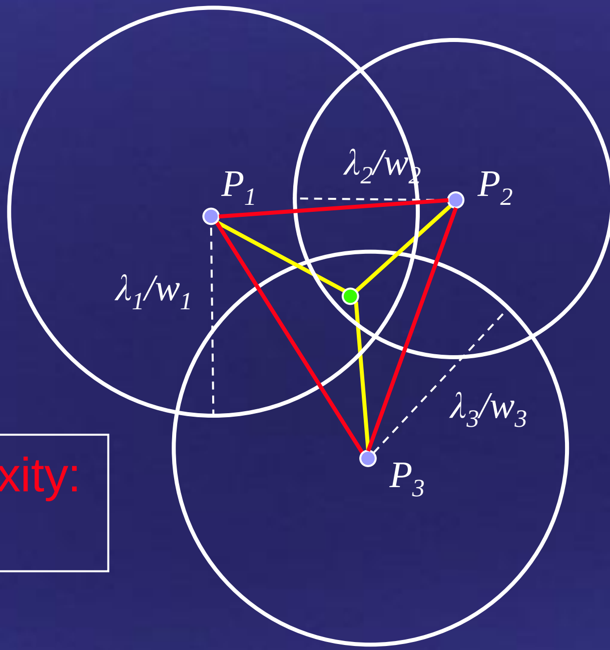
Computational complexity:

$O(|C| \log |C|)$

$$r_C = \sum_{i|P_i \in C} w_i D(P_i, S^*) - \sum_{i|P_i \in C} \lambda_i + \mu \geq$$

$$\geq \min_x \left\{ \sum_{i|P_i \in C} w_i^x \left| x - x_i \right| \right\} + \min_y \left\{ \sum_{i|P_i \in C} w_i^y \left| y - y_i \right| \right\} - \sum_{i|P_i \in C} \lambda_i + \mu$$

Lower Bound 2



Computational complexity:
O(|C|^2)

$$r_C = \sum_{i\in C}\left(w_i\,\mathrm{D}\left(P_i, S^*\right) - \lambda_i\right) + \mu = \frac{1}{N-1}\sum_{i\in C}\sum_{j\in C,\, j>i}\left(w_i\,\mathrm{D}\left(P_i, S^*\right) + w_j\,\mathrm{D}\left(P_j, S^*\right)\right) - \sum_{i\in C}\lambda_i + \mu$$

$$\geq \frac{1}{N-1}\sum_{i\in C}\sum_{j\in C,\, j>i}\min\{w_i, w_j\}\,\mathrm{D}\left(P_i, P_j\right) - \sum_{i\in C}\lambda_i + \mu$$

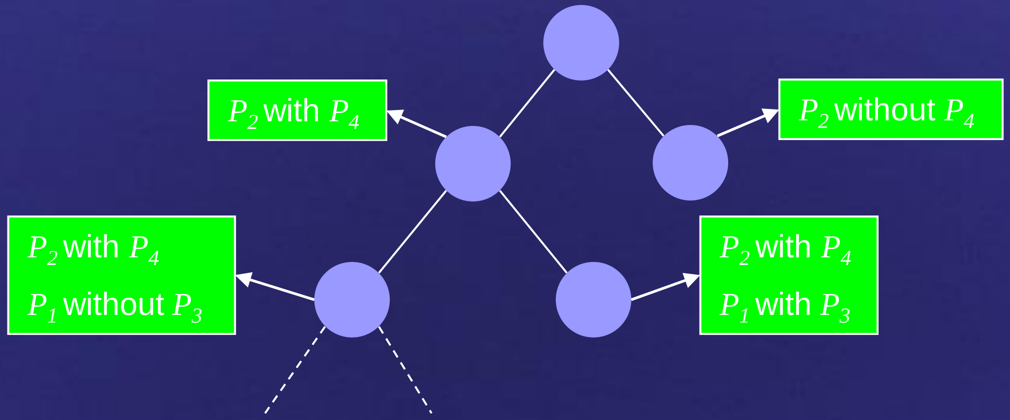
Clusters evaluation

- About 84% of clusters generated by DMW's algorithm are discarded, because LB1 or LB2 is non-negative.
- About 15% of clusters are discarded after their evaluation with Weiszfeld algorithm, because they have non-negative reduced cost.
- Only 1% of clusters are found to have negative reduced cost and are inserted into the RLMP.

Branching (1)

- If the optimal solution of RLMP is fractional one or more points are covered by more than one column.
- Let F be the subset of such points.
  For each P_u in F, Γ_u is the set of columns covering P_u and T_u is the set of points covered by the columns in Γ_u.
- Each pair of points (P_u, P_v) s.t. $P_v \epsilon T_u$
  is a candidate for a binary branching in which P_u and P_v are forced to belong to the same cluster in one branch and to different clusters in the other.

Branching (2)



- All clusters violating these *constraints* are recognized as soon as they are generated and they are discarded without being evaluated.

Branching (3)

- For each candidate branching pair we evaluate the unbalance between the two resulting children nodes.
- The unbalance is given by the difference (in modulo) between the sum of the values of the *z* variables corresponding to the columns which become infeasible.
- The branching pair (P_{u*}, P_{v*}) is the one for which such two sums are most balanced.

Branching (4)

$$X=\{\overbrace{\{P_1,P_2\}}^{z_1=0.6},\overbrace{\{P_1,P_5\}}^{z_2=0.75},\overbrace{\{P_1,P_4\}}^{z_3=0.65}\}$$

	c_1	c_2	c_3	Γ_1
P_1	1	1	1	1
P_2	1	0	0	1
P_3	0	0	0	0
P_4	0	0	1	1
P_5	0	1	0	1
	0.6	0.75	0.65	

$F=\{P_1\}$ $T_1=\{P_2, P_4, P_5\}$

$(P_1, P_2) = 0.75 + 0.65 - 0.6 = 0.8$

$(P_1, P_4) = 0.75 + 0.6 - 0.65 = 0.7$

$(P_1, P_5) = 0.65 + 0.6 - 0.75 = 0.5$

Branching pair = (P_1, P_5)

Experimental results (1)

Software configuration:

- ANSI C, Linux RedHat 9.

Hardware configuration:

- Athlon AMD Chipset 1.2 MHz with 768 MByte RAM.

Simplex solver:

- ILOG CPLEX 8.1

Time-out:

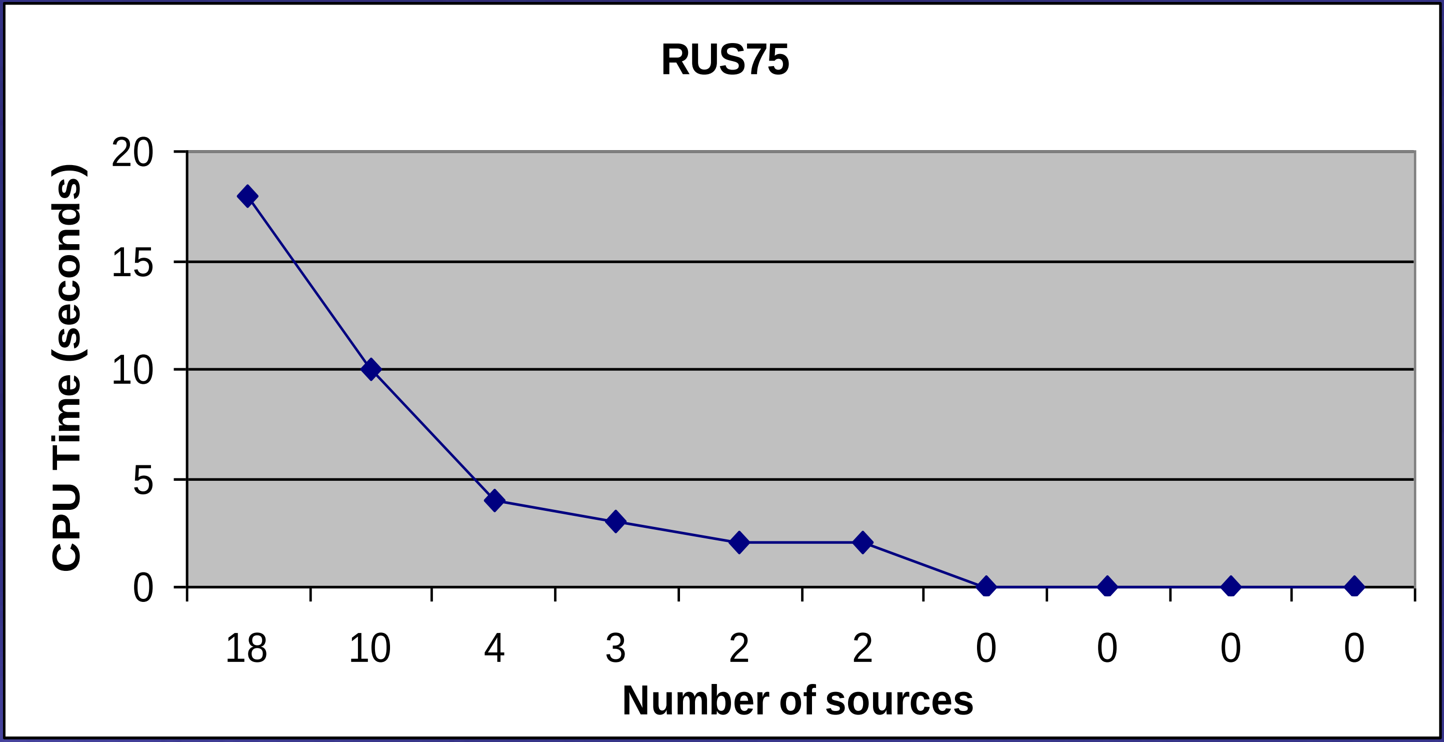
- 6 hours

Experimental results (2)

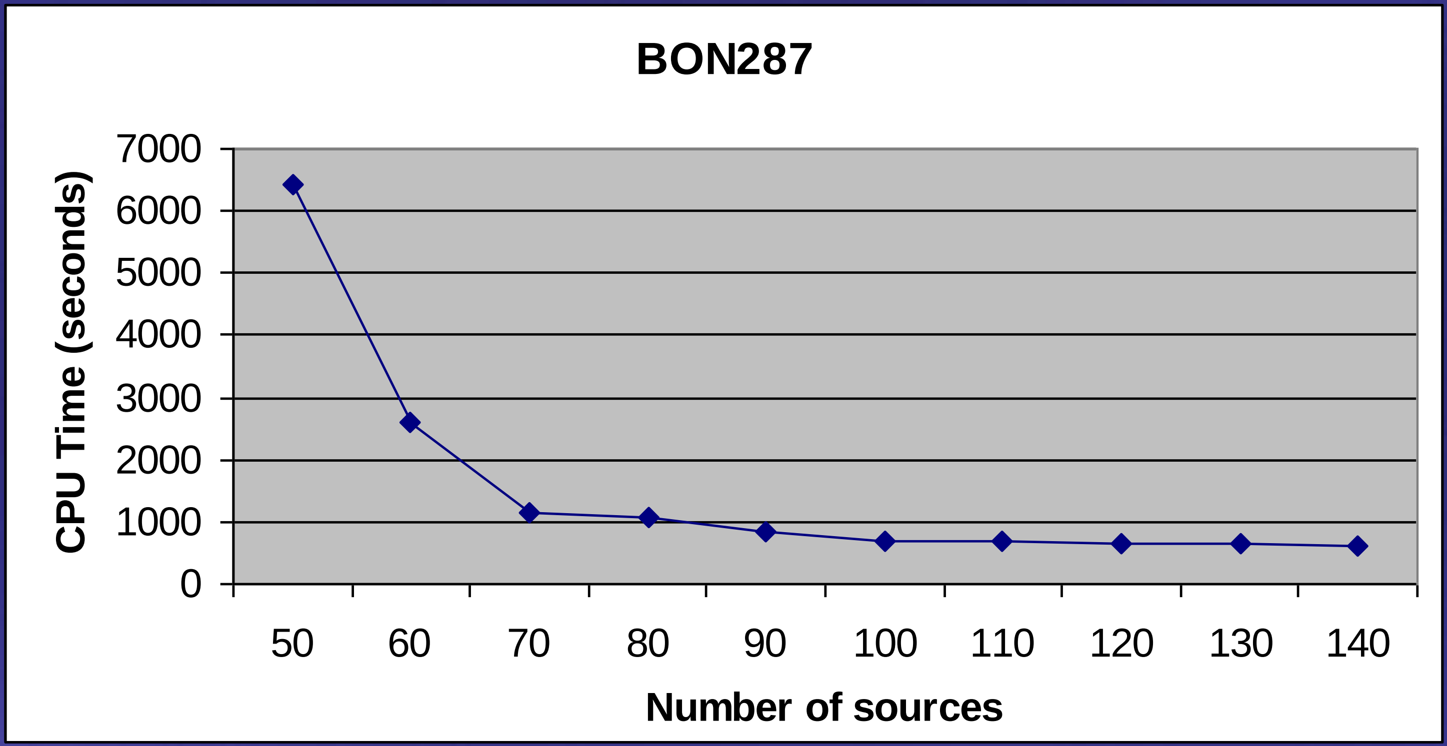
<u>*Dataset:*</u>

- *RUS75:* 75 points (Ruspini, 1970).
- *BON287:* 287 points (Bongartz et al., 1994). *This is a weighted dataset*.
- *REI654:* 654 points (Reinelt, 1991).
- *REI1060:* 1060 points (Reinelt, 1991).
- *RAND2000:* 2000 points, generated at random.

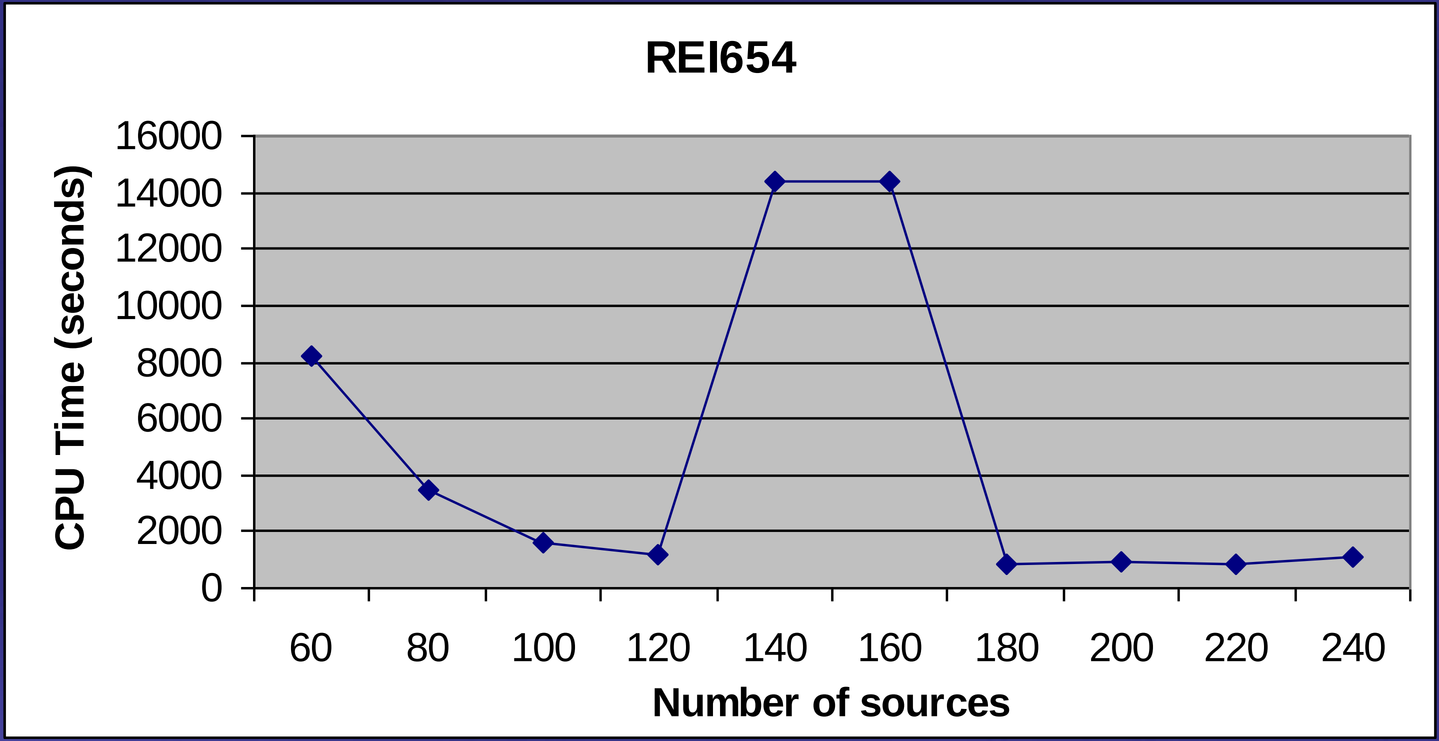
Experimental results (3)



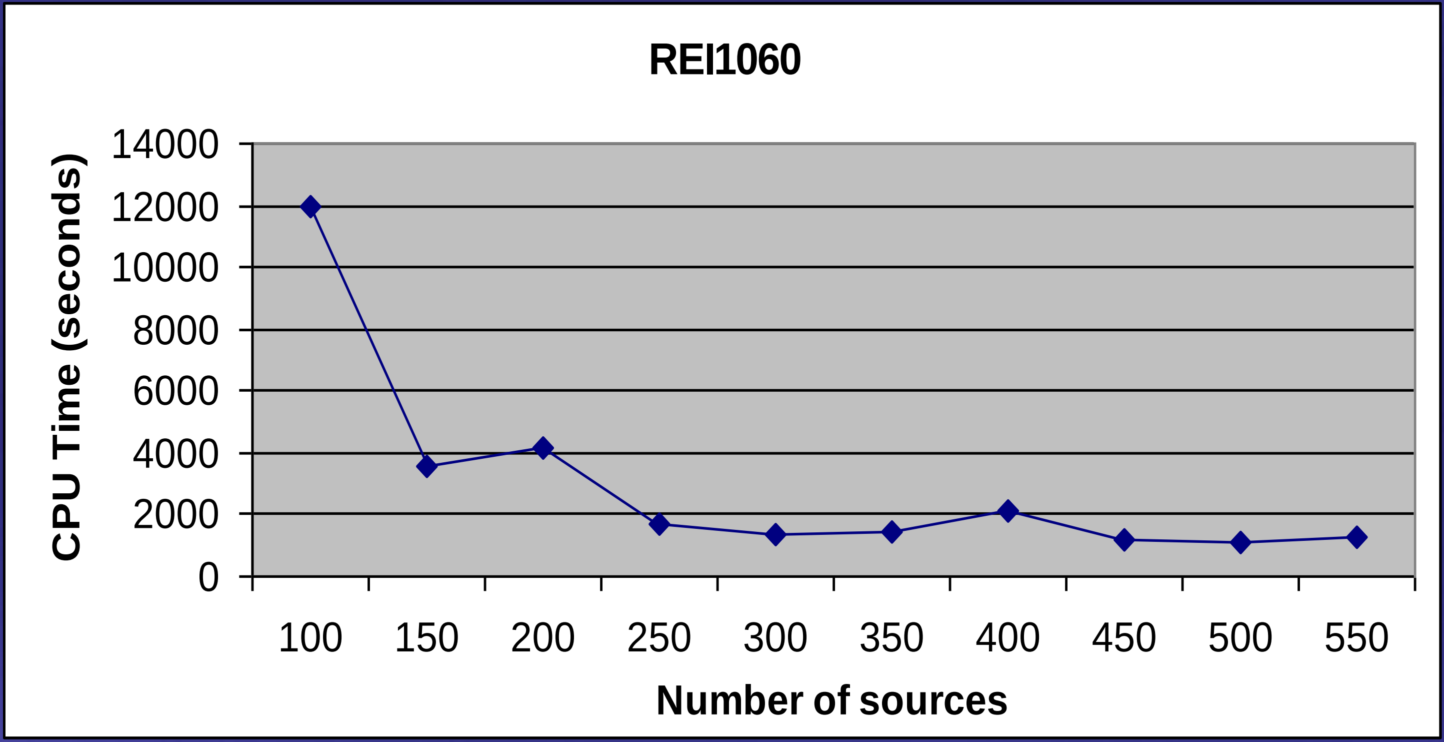
Experimental results (4)



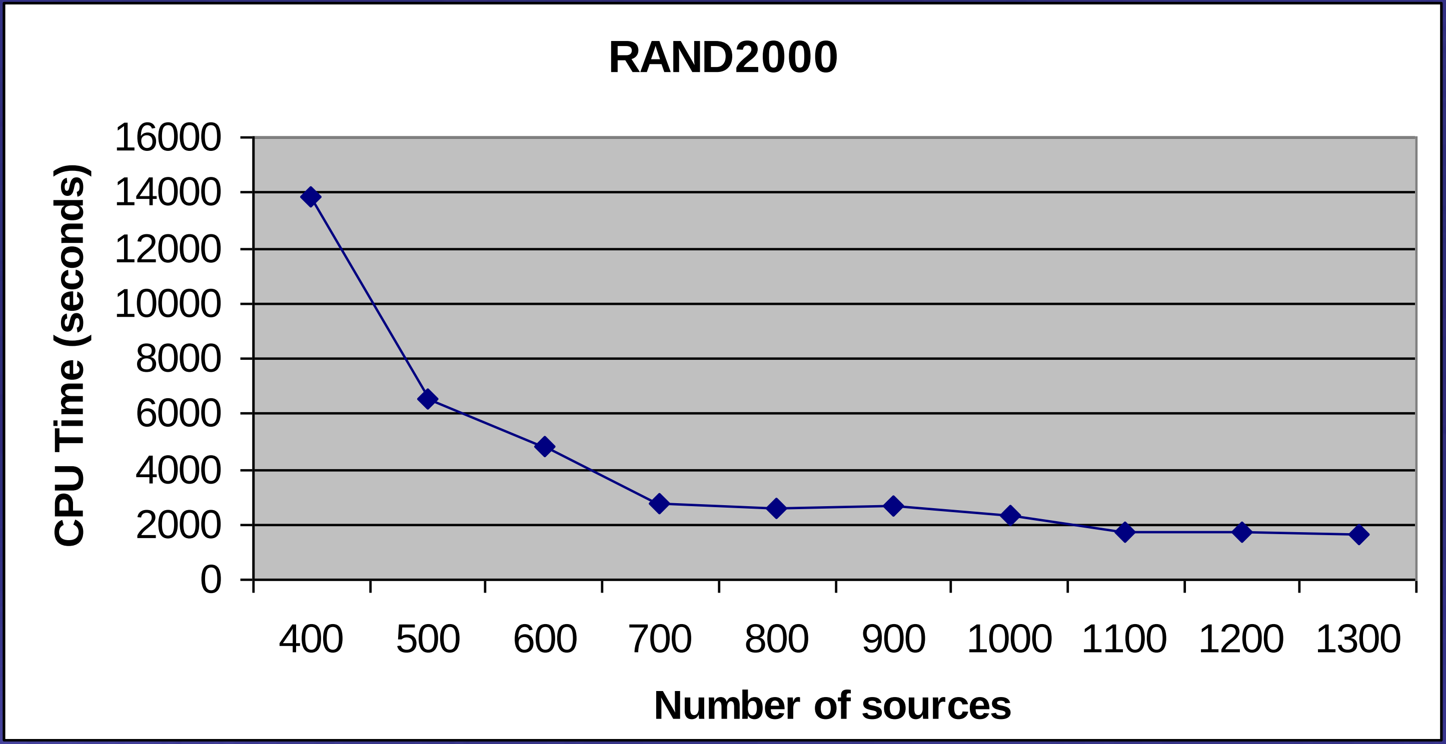
Experimental results (5)



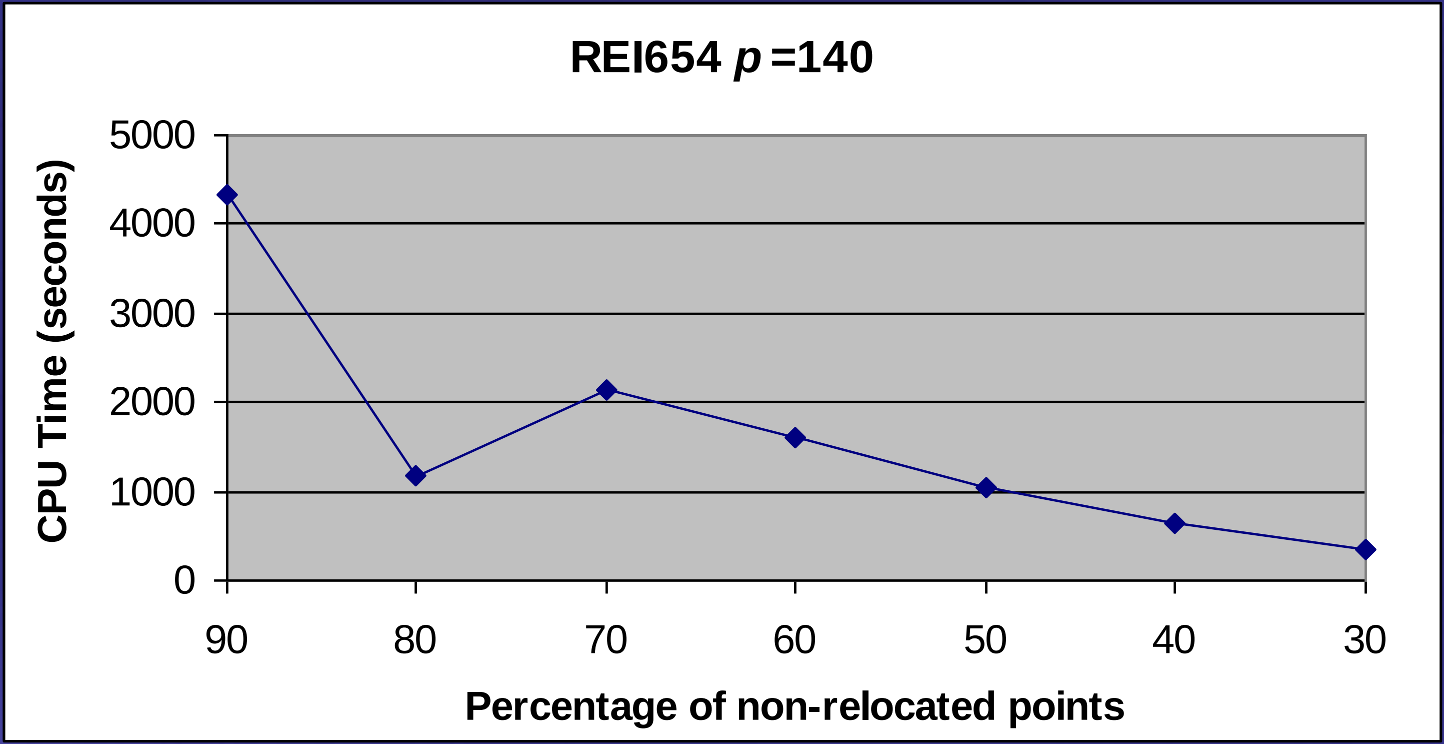
Experimental results (6)



Experimental results (7)



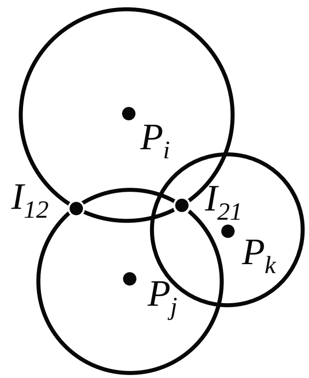
Experimental results (8)



Conclusions

- The algorithm performances do not depend only on the size of the instance but on its structure: clustered, regular, unweighted instances are more difficult than random ones.

- For each given set of points instances with more sources (and smaller clusters) are easier for our approach.
  This feature is complementary to the results of Krau reported by du Merle et al. (1999).



P_i
I_{12}
I_{21}
P_k
P_j

Pricing (1)